Angular Momentum Analysis of Spin $\frac{1}{2} \times \frac{1}{2}$ Scattering including Singlet-Triplet Mixing

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The angular momentum (partial wave) reduction of the Lippmann–Schwinger equation describing the interaction of two spin $\frac{1}{2}$ particles is extended to the case in which the spin singlet and triplet states are coupled. A straight forward method for obtaining the angular momentum decomposition of the general potential is indicated. The derived formalism is needed to describe the interaction between two nonidentical spin–$\frac{1}{2}$ particles or between two nucleons when isospin symmetry is violated. The resulting modification of the Stapp phase–shift analysis is given.

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I. INTRODUCTION

Recent experimental advances are permitting detailed measurements and analyses of the scattering of polarized nucleons from polarized nuclei [1]. This, in turn, is leading to a rigorous test of the theory describing spin-spin scattering [2]. The general form of the elastic scattering T matrix for two spin $\frac{1}{2}$ particles can be quite complicated. It simplifies considerably if we assume rotation invariance, parity conservation, and time reversal invariance, in which case [3–7]:

$$2T(\vec{k'},\vec{k}) = a(\vec{k'},\vec{k}) + b(\vec{k'},\vec{k}) + (a-b)\vec{\sigma}_n^p\vec{\sigma}_n^t + \Big(c(\vec{k'},\vec{k}) + d(\vec{k'},\vec{k})\Big)\,\vec{\sigma}_m^p\vec{\sigma}_m^t$$
$$+(c-d)\vec{\sigma}_l^p\vec{\sigma}_l^t + e(\vec{k'},\vec{k})(\vec{\sigma}_n^p + \vec{\sigma}_n^t) + f(\vec{k'},\vec{k})(\vec{\sigma}_n^p - \vec{\sigma}_n^t) \qquad (1)$$

Here $\vec{k}$ and $\vec{k'}$ are the initial and final momenta in the center–of–mass system, $\vec{\sigma^p}$ and $\vec{\sigma^t}$ are the Pauli spin operators for the projectile and target respectively, and the subscript on each $\vec{\sigma}$ indicates a dot products with one of the three independent unit vectors:

$$\hat{n} = \frac{\vec{k}\times\vec{k'}}{|\vec{k}\times\vec{k'}|}, \quad \hat{m} = \frac{\vec{k'}-\vec{k}}{|\vec{k'}-\vec{k}|}, \quad \hat{l} = \frac{\vec{k}+\vec{k'}}{|\vec{k}+\vec{k'}|} \qquad (2)$$

For example, $\vec{\sigma}_m^t = \vec{\sigma}^t\cdot\hat{m}$. For experimental (on-energy-shell) scattering, the $a-f$ coefficients in (1) are complex functions of the energy and scattering angle. Once these coefficients are known, all 36 possible experimental spin observables can be calculated from them [3,4].

Just as (1) is the most general scattering amplitude expressed as an operator in the direct product spin spaces of the projectile and target, so the projectile-target potential must have the same form:

$$2V(\vec{k'},\vec{k}) = V_{a+b}(\vec{k'},\vec{k}) + V_{a-b}(\vec{k'},\vec{k})\vec{\sigma}_n^p\vec{\sigma}_n^t + V_{c+d}(\vec{k'},\vec{k})\vec{\sigma}_m^p\vec{\sigma}_m^t$$
$$+V_{c-d}(\vec{k'},\vec{k})\vec{\sigma}_l^p\vec{\sigma}_l^t + V_e(\vec{k'},\vec{k})(\vec{\sigma}_n^p + \vec{\sigma}_n^t) + V_f(\vec{k'},\vec{k})(\vec{\sigma}_n^p - \vec{\sigma}_n^t) \qquad (3)$$

where the subscripts on V indicate the correspondence with (1). If the potential has the structure (3), then its use in a wave equation will generate a scattering amplitude of the form (1). In traditional language, the V_{a+b} term is a central potential, the V_{a-b}, V_{c+d}, and

V_{c-d} terms are tensor forces (dipole-dipole interactions for two electrons), the V_e term is the usual spin-orbit potential [5], and the V_f term is the unusual spin-orbit potential which couples spin singlet and triplet states.

Prior angular momentum (partial wave) analyses, such as those found in Goldberger and Watson [5], Stapp [8] and Goddard [9], have examined the scattering of two spin–$\frac{1}{2}$ particles for the nucleon-nucleon case where the f amplitude vanishes due to the isospin symmetry. In our work on proton scattering from spin–$\frac{1}{2}$ nuclei, we have had to develop a Stapp–like partial wave analysis [10] for cases in which the f term does not vanish (extensions of the analysis of observables including an f amplitude had already been given by La France and Winternitz [3]). While the work of Gersten [7] provides a partial wave analysis, it is in the helicity—not angular momentum— representation. While we suspect that others have deduced the general angular momentum analysis in the course of their research, we have been unable to find those analyses in the literature and so wish to present ours. Explicitly, we shall derive the partial–wave decomposition of the Lippmann–Schwinger equation when there is mixing between the spin singlet and triplet states (as well as within the triplet), and indicate how we have applied the method in our calculations.

II. ANGULAR MOMENTUM ANALYSIS WITH SINGLET-TRIPLET MIXING

Many–body effects and relativity leads to a potential V incorporating complicated non-localities. For this reason, rather than solve a Schrödinger equation we solve the equivalent, momentum space Lippmann–Schwinger equation:

$$T(\vec{k}',\vec{k}) = V(\vec{k}',\vec{k}) + \int \frac{d^3p}{E^+ - E(p)}\, V(\vec{k}',\vec{p})\, T(\vec{p},\vec{k}) \tag{4}$$

where $E(p) = E_p(p) + E_t(p)$ is the projectile plus target energy and the $+$ superscript indicates a positive $i\epsilon$ has been added to the on–shell energy $E = E_p(k_0) + E_t(k_0)$. Since the solution of a three–dimensional integral equation is a bit too formidable, we reduce it to coupled one–dimensional equations by expanding T and V in partial waves [10]:

$$V(\vec{k'},\vec{k}) = \frac{2}{\pi} \sum_{jm_j ll' ss'} i^{(l'-l)}\, V_{l'l}^{j(s's)}(k',k)\, \mathcal{Y}_{l's'}^{jm_j}(\hat{k'})\, \mathcal{Y}_{ls}^{\dagger jm_j}(\hat{k}) \tag{5}$$

$$T(\vec{k'},\vec{k}) = \frac{2}{\pi} \sum_{jm_j ll' ss'} i^{(l'-l)}\, T_{l'l}^{j(s's)}(k',k)\, \mathcal{Y}_{l's'}^{jm_j}(\hat{k'})\, \mathcal{Y}_{ls}^{\dagger jm_j}(\hat{k}) \tag{6}$$

Here l and s are the total orbital angular momentum and total spin for the target plus projectile, and j is the total angular momentum:

$$\vec{j} = \vec{l} + \vec{s} \tag{7}$$

$$\vec{s} = \frac{1}{2}(\vec{\sigma^t} + \vec{\sigma^p}), \quad s = 0(s), 1(t) \tag{8}$$

At times we denote the $s = 0$, singlet state with "s" and the $s = 1$, triplet state with "t".

The $\mathcal{Y}$'s in (5)-(6) are spin–angle functions and are given in our conventions [11] as:

$$\mathcal{Y}_{ls}^{jm}(\hat{k}) = \sum_{m_s m_l} \langle l m_l s m_s | jm \rangle\, Y_l^{m_l}(\theta,\phi)\, |s m_s\rangle \tag{9}$$

$$Y_l^m(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\, P_l^m(x \equiv \cos\theta) e^{im\phi} \tag{10}$$

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m P_l(x)}{dx^m} \tag{11}$$

To express the Lippmann–Schwinger equation (4) in the partial wave basis, we substitute (5) and (6), the partial–wave expansions of T and V, into (4) and employ the orthogonality of the spin-angle functions,

$$\int \mathcal{Y}_{ls}^{\dagger jm}(\hat{k})\, \mathcal{Y}_{l's'}^{jm'}(\hat{k})\, d\hat{k} = \delta_{ll'}\delta_{ss'}\delta_{mm'} \tag{12}$$

There results:

$$T_{l'l}^{j(s's)}(k',k) = V_{l'l}^{j(s's)}(k',k) + \frac{2}{\pi} \sum_{L\,S} \int_0^\infty p^2\, dp \frac{V_{lL}^{j(s'S)}(k',p) T_{Ll'}^{j(Ss)}(p,k)}{E^+ - E(p)} \tag{13}$$

In equation (13), the sum over $L\,S$ is over the orbital angular momenta and spin states which are allowed to couple for a fixed value of the total angular momentum j. Accordingly, the integral equations to solve in the partial wave basis are (we leave off the (k',k) dependence of the leftmost T's and V's):

$$\begin{bmatrix} T_{jj}^{j(ss)} \\ T_{jj}^{j(ts)} \end{bmatrix} = \begin{bmatrix} V_{jj}^{j(ss)} \\ V_{jj}^{j(ts)} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V_{jj}^{j(ss)}(k'p)\ V_{jj}^{j(st)}(k',p) \\ V_{jj}^{j(ts)}(k',p)\ V_{jj}^{j(tt)}(k',p) \end{bmatrix} \begin{bmatrix} T_{jj}^{j(ss)}(p,k) \\ T_{jj}^{j(ts)}(p,k) \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} T_{jj}^{j(tt)} \\ T_{jj}^{j(st)} \end{bmatrix} = \begin{bmatrix} V_{jj}^{j(tt)} \\ V_{jj}^{j(st)} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V_{jj}^{j(tt)}(k',p)\ V_{jj}^{j(ts)}(k',p) \\ V_{jj}^{j(st)}(k',p)\ V_{jj}^{j(ss)}(k',p) \end{bmatrix} \begin{bmatrix} T_{jj}^{j(tt)}(p,k) \\ T_{jj}^{j(st)}(p,k) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} T_{j-1\,j-1}^{j(tt)} \\ T_{j+1\,j-1}^{j(tt)} \end{bmatrix} = \begin{bmatrix} V_{j-1\,j-1}^{j(tt)} \\ V_{j+1\,j-1}^{j(tt)} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V_{j-1\,j-1}^{j(tt)}(k',p)\ V_{j-1\,j+1}^{j(tt)}(k',p) \\ V_{j+1\,j-1}^{j(tt)}(k',p)\ V_{j+1\,j+1}^{j(tt)}(k',p) \end{bmatrix} \begin{bmatrix} T_{j-1\,j-1}^{j(tt)}(p,k) \\ T_{j+1\,j-1}^{j(tt)}(p,k) \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} T_{j+1\,j+1}^{j(tt)} \\ T_{j-1\,j+1}^{j(tt)} \end{bmatrix} = \begin{bmatrix} V_{j+1\,j+1}^{j(tt)} \\ V_{j-1\,j+1}^{j(tt)} \end{bmatrix} + \int_0^\infty \frac{p^2 dp}{E^+ - E(p)} \begin{bmatrix} V_{j+1\,j+1}^{j(tt)}(k',p)\ V_{j+1\,j-1}^{j(tt)}(k',p) \\ V_{j-1\,j+1}^{j(tt)}(k',p)\ V_{j-1\,j-1}^{j(tt)}(k',p) \end{bmatrix} \begin{bmatrix} T_{j+1\,j+1}^{j(tt)}(p,k) \\ T_{j-1\,j+1}^{j(tt)}(p,k) \end{bmatrix} \tag{17}$$

Equations (14)-(15) describe spin singlet-triplet coupling arising from the V_f term in the potential (3) which in turn will produce an f amplitude in T (1). Equations (16)-(17) describe coupling within the spin triplet state arising from the tensor force terms $V_{a-b}, V_{c+d}, V_{c-d}$ in the potential (3), it mixes orbital angular momentum states [12]. Since the total angular momentum j is a conserved quantity, all coupled states have the same j superscript.

III. EVALUATION OF $V_{L'L}^{J(S'S)}$

We use a two–step procedure to determine the potential matrix elements $V_{l'l}^{j(s's)}(k',k)$ needed in (14)-(17). First we evaluate the potential (3) in the spin basis $|s, m_s\rangle$ and then we invert the angular momentum decomposition of these spin–basis potentials. The spin matrix elements required for the $V_a - V_e$ terms of (1) are given in Tables 7.1-7.4 of Goldberger and Watson [5]. As expected, the $V_a - V_e$ terms have vanishing matrix elements between singlet and triplet states. The new term V_f has nonvanishing matrix elements only between singlet and triplet states, explicitly

$$\langle 0,0|(\vec{\sigma}_n^p - \vec{\sigma}_n^t)|1,1\rangle \;= \langle 0,0|(\vec{\sigma}_n^p - \vec{\sigma}_n^t)|1,-1\rangle \;= -i\sqrt{2} \tag{18}$$

There is no θ or ϕ dependence in (18) because we adopt the "Madison Convention" shown in Figure 1. This convention takes the z-axis as the beam direction $\vec{k}$ ($\phi_i = \theta_i = 0$), and places the scattered momentum $\vec{k'}$ in the xz plane ($\theta_f = \theta, \phi_f = 0$). Accordingly:

$$\begin{aligned}
V_{ss}(\vec{k'},\vec{k}) &\equiv \langle 0,0|V|0,0\rangle \\
&= V_{a+b}(\vec{k'},\vec{k}) - V_{a-b}(\vec{k'},\vec{k}) - V_{c+d}(\vec{k'},\vec{k}) - V_{c-d}(\vec{k'},\vec{k}) && (19)\\
V_{s1}(\vec{k'},\vec{k}) &\equiv \langle 0,0|V|1,1\rangle = -V_{1s}(\vec{k'},\vec{k}) = V_{s-1}(\vec{k'},\vec{k}) \;=\; \frac{-i}{\sqrt{2}} V_f(\vec{k'},\vec{k}) && (20)\\
V_{00}(\vec{k'},\vec{k}) &\equiv \langle 1,0|V|1,0\rangle \\
&= V_{a+b}(\vec{k'},\vec{k}) + V_{a-b}(\vec{k'},\vec{k}) + \Big(V_{c+d}(\vec{k'},\vec{k}) - V_{c-d}(\vec{k'},\vec{k})\Big)\cos\theta && (21)\\
V_{11}(\vec{k'},\vec{k}) &= V_{-1-1} = V_{a+b}(\vec{k'},\vec{k}) + V_{c+d}(\vec{k'},\vec{k})\sin^2\frac{\theta}{2} + V_{c-d}(\vec{k'},\vec{k})\cos^2\frac{\theta}{2} && (22)\\
V_{10}(\vec{k'},\vec{k}) &= -V_{-10} \\
&= \frac{-i}{\sqrt{2}} V_e(\vec{k'},\vec{k}) - \frac{1}{\sqrt{2}} V_{c+d}(\vec{k'},\vec{k})\sin\theta + \frac{1}{\sqrt{2}} V_{c-d}(\vec{k'},\vec{k})\sin\theta && (23)\\
V_{01}(\vec{k'},\vec{k}) &= -V_{0-1} \\
&= \frac{i}{\sqrt{2}} V_e(\vec{k'},\vec{k}) - \frac{1}{\sqrt{2}} V_{c+d}(\vec{k'},\vec{k})\sin\theta + \frac{1}{\sqrt{2}} V_{c-d}(\vec{k'},\vec{k})\sin\theta && (24)\\
V_{1,-1}(\vec{k'},\vec{k}) &= V_{-11} = -V_{a-b}(\vec{k'},\vec{k}) + V_{c+d}(\vec{k'},\vec{k})\cos^2\frac{\theta}{2} + V_{c-d}(\vec{k'},\vec{k})\sin^2\frac{\theta}{2} && (25)
\end{aligned}$$

Here we have the matrix elements of V in the spin basis. We next expand these matrix elements in angular momentum states in order to determine the partial–wave matrix elements $V_{l'l}^{j(s's)}$. We take the expansion of the potential in spin–angle functions (5) and evaluate the matrix element between spin states:

$$\begin{aligned}
\langle s'''m_s'''|V(\vec{k'},\vec{k})|s''m_s''\rangle = \frac{2}{\pi} \sum_{js'sl'lm_sm_{s'}m_lm_{l'}} & \langle s'''m_s'''|s'm_s'\rangle\, i^{l-l'}\, \langle l'm_l's'm_s'|jm\rangle \\
\times \langle jm|lm_lsm_s\rangle \quad & Y_{l'}^{m_{l'}}(\theta_f,\phi_f)\, V_{l'l}^{js's}(k',k)\, Y_l^{m_l^*}(\theta_i,\phi_i)\, \langle sm_s|s'',m_s''\rangle
\end{aligned} \tag{26}$$

where (k,θ_i,ϕ_i) and (k',θ_f,ϕ_f) are the spherical coordinates of the initial and final momenta. The Clebsch-Gordon coefficients vanish unless $l, l' = j \pm 1, j$ and $m_j = m_l + m_s = m_l' + m_s'$. Parity conservation requires $|l - l'| = 0, 2$. In the "Madison Convention" (Figure 1) the

projectile has no angular momentum in its propagation direction and so $m_l = 0$, in which case

$$Y_l^{m_l^*}(\theta_i, \phi_i) = Y_l^0(0,0) = \sqrt{\frac{2l+1}{4\pi}} \tag{27}$$

As a sample, we concentrate on the new term V_{s1} (20) which couples the $|00\rangle$ singlet state to the $|11\rangle$ triplet state. Because j is a constant and $s' = 0$ in the final state, the total angular momentum j must equal l'. The parity constraint then requires that $l = l'$. Because m_s and m_l are 1 and 0, respectively, we deduce that $m_j = 1$, and $m_{l'} = 1$. For this V_{s1} term, the sum in (26) reduces to a simple sum in the final orbital angular momentum l':

$$\sum_{js'sll'm_s m_{s'} m_l m_{l'}} ... \to \sum_l ... \tag{28}$$

Similarly, when the spin-dependent singlet-singlet potential $V_{ss}(\vec{k'}, \vec{k})$ and those within the triplet state $V_{m'm}(\vec{k'}, \vec{k})$ are evaluated, we obtain the desired partial–wave expansion of V (and of T with the interchange $V \to T$):

$$V_{s1}(\vec{k'}, \vec{k}) = \frac{-\sqrt{2}}{4\pi^2} \sum_{l=1} P_l^1(x = \cos\theta_{k'k}) \frac{2l+1}{\sqrt{l(l+1)}} V_{ll}^{l(st)}(k', k) \tag{29}$$

$$V_{ss}(\vec{k'}, \vec{k}) = \frac{1}{2\pi^2} \sum_{l=0} P_l(x)(2l+1) V_{ll}^{l(ss)}(k', k) \tag{30}$$

$$\begin{aligned} V_{11}(\vec{k'}, \vec{k}) = \frac{1}{4\pi^2} \sum_{l=0} P_l(x) \Big\{ & (l+2) V_{ll}^{l+1(tt)}(k', k) - \sqrt{(l+1)(l+2)}\, V_{l\,l+2}^{l+1(tt)}(k', k) \\ & + (2l+1) V_{ll}^{l(tt)}(k', k) + (l-1) V_{ll}^{l-1(tt)}(k', k) - \sqrt{(l-1)l}\, V_{l\,l-2}^{l-1(tt)}(k', k) \Big\} \end{aligned} \tag{31}$$

$$\begin{aligned} V_{00}(\vec{k'}, \vec{k}) = \frac{1}{2\pi^2} \sum_{l=0} P_l(x) \Big\{ & (l+1) V_{ll}^{l+1(tt)}(k', k) + l V_{ll}^{l-1(tt)}(k', k) \\ & + \sqrt{(l+1)(l+2)}\, V_{l\,l+2}^{l+1(tt)}(k', k) + \sqrt{(l-1)l}\, V_{l\,l-2}^{l-1(tt)}(k', k) \Big\} \end{aligned} \tag{32}$$

$$\begin{aligned} V_{10}(\vec{k'}, \vec{k}) = \frac{\sqrt{2}}{4\pi^2} \sum_{l=1} P_l^1(x) \Big\{ & -V_{ll}^{l-1(tt)}(k', k) + V_{ll}^{l+1(tt)}(k', k) \\ & + \sqrt{\frac{l+2}{l+1}} V_{l\,l+2}^{l+1(tt)}(k', k) - \sqrt{\frac{l-1}{l}} V_{l\,l-2}^{l-1(tt)}(k', k) \Big\} \end{aligned} \tag{33}$$

$$\begin{aligned} V_{01}(\vec{k'}, \vec{k}) = \frac{\sqrt{2}}{4\pi^2} \sum_{l=1} P_l^1(x) \Big\{ & -\frac{l+2}{l+1} V_{ll}^{l+1(tt)}(k', k) + \frac{2l+1}{l(l+1)} V_{ll}^{l(tt)}(k', k) \\ & + \frac{l-1}{l} V_{ll}^{l-1(tt)}(k', k) + \sqrt{\frac{l+2}{l+1}} V_{l\,l+2}^{l+1(tt)}(k', k) - \sqrt{\frac{l-1}{l}} V_{l\,l-2}^{l-1(tt)}(k', k) \Big\} \end{aligned} \tag{34}$$

$$V_{1\,-1}(\vec{k'},\vec{k}) = \frac{1}{4\pi^2}\sum_{l=2} P_l^2(x)\left\{\frac{1}{l+1}V_{ll}^{l+1(tt)}(k',k) - \frac{1}{\sqrt{(l+1)(l+2)}}V_{ll+2}^{l+1(tt)}(k',k)\right.$$
$$\left. -\frac{2l+1}{l(l+1)}V_{ll}^{l(tt)}(k',k) + \frac{1}{l}V_{ll}^{l-1(tt)}(k',k) - \frac{1}{\sqrt{l(l-1)}}V_{ll-2}^{l-1(tt)}(k',k)\right\} \tag{35}$$

Note that the sum is actually over the orbital angular momentum l' of the final state, but for notational simplicity we have changed the label l' to l. Furthermore, note that the organization in (29)-(35) combines matrix elements which multiply the same Legendre polynomial even though the matrix elements may correspond to different j values. The Lippmann–Schwinger equations (14)-(17) of course only couple states with the same j.

We invert equations (29)-(35) for the partial wave potentials $V_{l'l}^{j(s's)}(k',k)$ by projections based on the orthogonality of the associated Legendre polynomials. Specifically, we multiply the equation for each $V_{m'm}$ by $P_l^{|m'-m|}$, and evaluate numerically the integral

$$I_{m'm}(k',k) = \int_{-1}^{1} dx\, V_{m'm}(\vec{k'},\vec{k})\, P_l^{|m'-m|}(\cos\theta_{k'k}) \tag{36}$$

For (29) and (30) the inversion is simple because only one $V_{l'l}^{j(s's)}$ is involved:

$$V_{ll}^{l(st)}(k',k) = V_{ll}^{l(ts)}(k',k) = \frac{-\sqrt{2}\pi^2}{\sqrt{l(l+1)}}I_{s1}(k',k) \tag{37}$$

$$V_{ll}^{l(ss)}(k',k) = \pi^2 I_{ss}(k',k) \tag{38}$$

The equations (31)-(35) contain $V_{l'l}^{j(tt)}$'s intermixed for differing j and l values, and so the projection results in five coupled equations in five unknowns:

$$\vec{I} = B\vec{V} \tag{39}$$

$$\begin{bmatrix} I_{11}(k',k) \\ I_{00}(k',k) \\ I_{10}(k',k) \\ I_{01}(k',k) \\ I_{1-1}(k',k) \end{bmatrix} = \begin{bmatrix} \\ B_{l'l}^{m'm} \\ \\ \\ \end{bmatrix} \begin{bmatrix} V_{ll}^{l+1(tt)}(\vec{k'},\vec{k}) \\ V_{ll}^{l(tt)}(\vec{k'},\vec{k}) \\ V_{ll}^{l-1(tt)}(\vec{k'},\vec{k}) \\ V_{ll+2}^{l+1(tt)}(\vec{k'},\vec{k}) \\ V_{ll-2}^{l-1(tt)}(\vec{k'},\vec{k}) \end{bmatrix} \tag{40}$$

where $[B^{m'm}_{l'l}]$ is the matrix of coefficients multiplying the V's in (31)-(35). In our application [13], the matrix equation (40) is solved by numerical inversion

$$\vec{V} = B^{-1}\vec{I} \tag{41}$$

and then the whole procedure is checked by recombining the potential according to (29)-(35).

Once the $V^{j(s's)}_{l'l}$'s are known, the coupled integral equations (14)-(17) are solved numerically for the $T^{j(s's)}_{l'l}$'s using a modification of the Haftel–Tabakin technique [12]. Once the $T^{j(s's)}_{l'l}$'s are known, the matrix elements in the spin basis $\langle s'm_{s'}|T|sm_s\rangle$ are computed via the generalization of equations (29)-(35) in which the V's are replaced by T's. Finally, once the $\langle s'm_{s'}|T|sm_s\rangle$ are known, the $a-f$ amplitudes of (1) are obtained [3]:

$$a(\vec{k'},\vec{k}) = \frac{1}{2}\left(T_{11}(\vec{k'},\vec{k}) + T_{00}(\vec{k'},\vec{k}) - T_{1-1}(\vec{k'},\vec{k})\right) \tag{42}$$

$$b(\vec{k'},\vec{k}) = \frac{1}{2}\left(T_{11}(\vec{k'},\vec{k}) + T_{ss}(\vec{k'},\vec{k}) + T_{1-1}(\vec{k'},\vec{k})\right) \tag{43}$$

$$c(\vec{k'},\vec{k}) = \frac{1}{2}\left(T_{11}(\vec{k'},\vec{k}) - T_{ss}(\vec{k'},\vec{k}) + T_{1-1}(\vec{k'},\vec{k})\right) \tag{44}$$

$$d(\vec{k'},\vec{k}) = \frac{1}{2}\left(T_{00}(\vec{k'},\vec{k}) + T_{1-1}(\vec{k'},\vec{k}) - T_{11}(\vec{k'},\vec{k})\right)/2\cos\theta_{k'k} \tag{45}$$

$$= -\left(T_{10}(\vec{k'},\vec{k}) + T_{01}(\vec{k'},\vec{k})\right)/\sqrt{2}\sin\theta_{k'k} \tag{46}$$

$$e(\vec{k'},\vec{k}) = \frac{i}{\sqrt{2}}\left(T_{10}(\vec{k'},\vec{k}) - T_{01}(\vec{k'},\vec{k})\right) \tag{47}$$

$$f(\vec{k'},\vec{k}) = i\sqrt{2}T_{s1}(\vec{k'},\vec{k}) \tag{48}$$

All spin observables are then calculated from $a-f$ using the relations found in La France and Winternitz [3].

IV. RELATION TO PHASE SHIFTS

The on-energy-shell T matrix elements in the partial wave basis $T^{j(s's)}_{l'l}(k_0,k_0)$ can be related to phase shifts, a convenient phenomenological parameterization of the scattering data. This is particularly useful when the present formalism is applied to the two nucleon problem because there are tables of NN phase shifts. The relations to the bar phases are [7,8,12]:

$$-2i\rho T^{j(ss)}_{j\,j}(k_0,k_0) = \cos 2\bar{\gamma}_l e^{2i\bar{\delta}_j} - 1 \quad (49)$$

$$-2i\rho T^{j(tt)}_{j\,j}(k_0,k_0) = \cos 2\bar{\gamma}_l e^{2i\bar{\delta}_{jj}} - 1 \quad (50)$$

$$-2i\rho T^{j\,(tt)}_{j\pm1\,j\pm1}(k_0,k_0) = \cos 2\bar{\epsilon}_j e^{2i\bar{\delta}_{j\pm1\,j}} - 1 \quad (51)$$

$$-2i\rho T^{j\,(tt)}_{j\pm1\,j\mp1}(k_0,k_0) = -i\sin 2\bar{\epsilon}_j\, e^{2i(\bar{\delta}_{j-1\,j}+\bar{\delta}_{j+1\,j})} \quad (52)$$

$$-2i\rho T^{j(ts)}_{j\,j}(k_0,k_0) = -i\sin 2\bar{\gamma}_l\, e^{i(\bar{\delta}_j+\bar{\delta}_{jj})} \quad (53)$$

$$\rho = 2k_0\frac{E_p(k_0)E_t(k_0)}{E_p(k_0)+E_t(k_0)} \quad (54)$$

The parameter $\bar{\gamma}_l$ is the mixing angle between the $|0,0\rangle$ singlet and $|1,1\rangle$ triplet state, and the parameter $\bar{\epsilon}_l$ is the mixing angle between the l and $l+2$ triplet states. In Table I we give the connection to the α notation of Stapp [8] as well as that to the N_{spin} notation used in our computer code *Lpotp2*.

V. SUMMARY

We have extended the partial wave analysis of the Schrödinger equation describing the interaction of two spin $\frac{1}{2}$ particles to the case where a potential couples the spin singlet and triplet states as well as coupling within the triplet state. In particular, we have concentrated on the scattering configuration described by the Lippmann–Schwinger equation although the same formalism can be used for bound states. While a previous formalism was appropriate to the nucleon-nucleon problem, extensions are necessary when the two Fermions are not identical. We are now applying the new formalism to describe polarized proton scattering from polarized ^3He and ^{13}C nuclei [14]. Our formalism should also be useful in describing the interaction between identical hadrons when isospin symmetry is violated. Furthermore, the formalism may find some applicability in the electron-atom interaction in atoms such as calcium where there is strong singlet-triplet mixing.

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[10] Note although neither T nor V is a wave, we refer to their angular momentum decomposition as a "partial–wave" expansion. Note too that in some previous work, $V_{ll}^{l(ss)}$ was called V_l^l for short.

[11] This is the sign convention of Messiah and of Goldberger and Watson. It differs by $(-1)^m$ from Abrahmowitz and Stegun, Jackson, and Ref. [9].

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FIGURES

FIG. 1. The coordinate system used to describe the scattering of momentum $\vec{k}$ into $\vec{k'}$ in the Madison convention. The incident momentum is along the z axis and the final momentum is in the xz plane. Note that θ' is the same as the $\theta_{k'k}$ in equations (29)-(35).

TABLES

TABLE I. Notations for spin $\frac{1}{2} \times \frac{1}{2}$ amplitudes ($T^{0(tt)}_{0\,0} = T^{0(tt)}_{-1\,-1} = T^{0(tt)}_{1\,-1} = T^{0(tt)}_{-1\,1} = 0$).

$T^{j(s's)}_{l'\,l}$	$T^{j(ss)}_{j\,j}$	$T^{j(ts)}_{j\,j}$	$T^{j(tt)}_{j-1\,j-1}$	$T^{j(tt)}_{j+1\,j-1}$	$T^{j(tt)}_{j+1\,j+1}$	$T^{j(tt)}_{j-1\,j+1}$	$T^{j(tt)}_{j\,j}$	$T^{j(st)}_{j\,j}$
Spin	0	$1 \leftarrow 0$	1	1	1	1	1	$0 \leftarrow 1$
Δl	0	0	0	2	0	-2	0	0
Stapp	α_l	–	$\alpha_{l,l+1}$	α^{l-1}	$\alpha_{l,l-1}$	α^{l+1}	α_{ll}	–
N_{spin}	1	2	3	4	5	6	7	8

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